Distribution of the largest aftershocks in branching models of triggered seismicity: Theory of the universal Båth law

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Using the epidemic-type aftershock sequence (ETAS) branching model of triggered seismicity, we apply the formalism of generating probability functions to calculate exactly the average difference between the magnitude of a mainshock and the magnitude of its largest aftershock over all generations. This average magnitude difference is found empirically to be independent of the mainshock magnitude and equal to 1.2, a universal behavior known as Båth's law. Our theory shows that Båth's law holds only sufficiently close to the critical regime of the ETAS branching process. Allowing for error bars ±0.1 for Båth's constant value around 1.2, our exact analytical treatment of Båth's law provides new constraints on the productivity exponent α and the branching ratio n: $0.9 \lesssim \alpha \leqslant 1$ and $0.8 \lesssim n \leqslant 1$. We propose a method for measuring α based on the predicted renormalization of the Gutenberg-Richter distribution of the magnitudes of the largest aftershock. We also introduce the "second Båth law for foreshocks:" the probability that a main earthquake turns out to be the foreshock does not depend on its magnitude ρ.

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I. INTRODUCTION

This paper is part of our continuing effort to develop a complete theory of seismicity within models of triggered seismicity, which allows one to make quantitative predictions of observables that can be compared with empirical data [1–3,6–8]. We study the general branching process, called the epidemic-type aftershock sequence (ETAS) model of triggered seismicity, introduced by Ogata in the present form [9] and by Kagan and Knopoff in a slightly different form [10] and whose main statistical properties are reviewed in [2]. The ETAS model belongs to a general class of branching processes [11,12], and has in addition the property that the variance of the number of earthquake progenies triggered in direct lineage from a given mother earthquake is mathematically infinite. This model has been shown to constitute a powerful null hypothesis to test against other models [9]. The advantage of the ETAS model is its conceptual simplicity based on three independent well-found empirical laws (Gutenberg-Richter distribution of earthquake magnitudes, Omori law of aftershocks, and productivity law) and its power of explanation of other empirical observations (see, for instance, [3] and references therein).

Here, we develop a theoretical formulation based on generating probability functions (GPF) to construct the distribution of magnitudes of the largest triggered event (largest aftershock) within the cascade comprising all triggered events of a given source earthquake. This allows us to derive the empirical Båth's law [13,14], which states that the average difference in magnitude between a mainshock and its largest aftershock is 1.2 regardless of the mainshock magnitude, within a completely consistent theory taking into account all generations of triggered events. Our present results significantly improve on the numerical results of [14] by demonstrating the essential roles played by the cascade of triggered events and the proximity to criticality in order to obtain Båth's law and by providing improved constraints on the key parameters of the ETAS model. In addition, we extend Båth's law, which is a statement on the average magnitude difference between the mainshock and its largest aftershock, by giving the full distribution. Our theoretical framework also allows us to calculate precisely the probability that the largest aftershock turns out to be larger than its source, a situation that is usually interpreted as the source and all events before the largest aftershock being its foreshocks, the largest aftershock being reinterpreted as the mainshock of the seismic series.

The paper is organized as follows. Section II recalls the definition of the branching model of triggered seismicity. Section III presents the generating probability function and results on the statistics of the largest aftershock among aftershocks of the first generation. The GPF for first-generation aftershocks is generalized to aftershocks of all generations in Sec. IV. This allows us to predict that the distribution of magnitudes of the largest aftershock over all aftershock generations is renormalized in the critical regime. This renormalization provides a way to calibrate the productivity parameter. Section V puts together previous results to calculate the average difference in magnitude between the mainshock and its largest aftershock over all generations. In the critical regime, Båth's law is shown to hold. The value of the aver-

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age difference in magnitude allows us to offer improved constraints on the two key parameters of the ETAS model; the critical branching ratio n and the productivity exponent α. Section VI concludes.

II. THE EPIDEMIC-TYPE AFTERSHOCK SEQUENCE BRANCHING MODEL OF EARTHQUAKES

Consider an earthquake of magnitude ρ, which we refer to as a mainshock, meaning that we are interested in the earthquakes that it triggers (aftershocks). According to the ETAS model, it generates a random number R_ρ^1 of first-generation aftershocks, which has Poissonian statistics,

$$P_\rho(R_\rho^1) = e^{-\kappa\mu}\frac{(\mu\kappa)^{R_\rho^1}}{(R_\rho^1)!}, \tag{1}$$

characterized by the conditional average number

$$N_\rho = \kappa\mu(\rho), \quad \mu(\rho) = 10^{\alpha(\rho-m_0)}. \tag{2}$$

Here m_0 is the minimum magnitude of earthquake capable of triggering other earthquakes, and κ is a constant. Expression (2) for $\mu(\rho)$ is chosen in such a way that it reproduces the empirical dependence of the average number of aftershocks triggered directly by an earthquake of magnitude m (see [15,16] and references therein). Expression (2) gives the so-called productivity law of a given mother as a function of its magnitude ρ.

The ETAS model requires the specification of the Gutenberg-Richter density distribution of earthquake magnitudes

$$p(m) = b\ln(10)10^{-b(m-m_0)}, \quad m \geqslant m_0, \tag{3}$$

such that $\int_m^\infty p(x)dx$ gives the probability that an earthquake has a magnitude equal to or larger than m. This magnitude distribution $p(m)$ is assumed to be independent of the magnitude of the triggering earthquake, i.e., a large earthquake can be triggered by a smaller one [3]. The cumulative $[\mathcal{P}(m)]$ and complementary cumulative $[\mathcal{Q}(m)]$ distributions corresponding to (3) are

$$\mathcal{P}(m) = 1 - \mathcal{Q}(m), \quad \mathcal{Q}(m) = 10^{-b(m-m_0)} = [\mu(m)]^{-\gamma},$$

$$\gamma = b/\alpha. \tag{4}$$

The ETAS model is defined by the conditional Poisson intensity given the average rate of seismicity at time t and position $\mathbf{r}$ conditioned on all past earthquakes

$$\lambda(t,\mathbf{r}) = s(t,\mathbf{r}) + \sum_{i|t_i\leqslant t}\mu(m_i)\Psi(t-t_i)\phi(\mathbf{r}-\mathbf{r}_i), \tag{5}$$

where $s(t,\mathbf{r})$ is the average Poisson rate of spontaneous earthquake sources ("immigrants," in the language of epidemic branching processes) at position $\mathbf{r}$ and at time t. The sum is over all past earthquakes: each earthquake is characterized by its occurrence time t_i, its magnitude m_i, and its location $\mathbf{r}_i$ in the catalog. The two kernels, $\Psi(t-t_i)$ and $\phi(\mathbf{r}-\mathbf{r}_i)$, whose integrals with respect to time and space, respectively, are normalized to 1, describe the contribution of the earthquake at time t_i and position $\mathbf{r}_i$ to the seismic intensity at time t in the future and at position $\mathbf{r}$.

III. STATISTICS OF THE LARGEST AFTERSHOCK AMONG AFTERSHOCKS OF THE FIRST GENERATION

A. Generating probability function of aftershocks of the first generation triggered by an arbitrary mainshock

The Poissonian statistics of the aftershocks of the first generation implies that the generating probability function of their numbers reads

$$\Theta_1(z|\rho) = \langle z^{R_\rho^1}\rangle = \sum_{r=0}^{\infty} P_\rho(r)z^r = e^{\kappa\mu(\rho)(z-1)}, \tag{6}$$

where we have used (1), and the angle brackets correspond to statistical averaging.

Let $M_1, M_2, \ldots, M_{R_\rho^1}$ be the random magnitudes of the R_ρ^1 aftershocks of the first generation triggered by the mainshock. Let us consider the statistical average defined by

$$\Theta_1(z,m|\rho) = \left\langle z^{R_\rho^1}\prod_{k=1}^{R_\rho^1} H(m-M_k)\right\rangle, \tag{7}$$

where H is the Heaviside function. Using the Poissonian statistics of the random number R_ρ^1 of aftershock numbers and the Gutenberg-Richter law for their magnitudes, we obtain

$$\Theta_1(z,m|\rho) = \sum_{r=0}^{\infty} P_\rho(r)z^r[\mathcal{P}(m)]^r = e^{\kappa\mu[z\mathcal{P}(m)-1]}. \tag{8}$$

Equation (8) gives the GPF of the number of first-generation aftershocks triggered by some source, whose magnitude is equal to ρ. The main difference between the GPF $\Theta_1(z,m|\rho)$ from the standard GPF $\Theta_1(z|\rho)$ given by (6) is that $\Theta_1(z,m|\rho)$ in (8) describes the statistics only of the sequences of first-generation aftershocks, whose magnitudes $\{M_1, M_2, \ldots, M_{R_1}\}$ are all smaller than some given magnitude m.

Notice that $\Theta_1(z,m|\rho)$ can be rewritten as

$$\Theta_1(z,m|\rho) = \langle z^{R_\rho^1(m)}H(m-M_\rho^1)\rangle, \tag{9}$$

where M_ρ^1 is the largest magnitude over all aftershocks of the first generation triggered by the mainshock and $R_\rho^1(m)$ is the number of aftershocks for those realizations of aftershocks in which no aftershocks of the first generation exceed the magnitude m. The transformation from (8) to (9) is an obvious generalization of the following relation of order statistics (see, for instance, [4]). Let us have $R_1 = r$ random values $\{M_1, M_2, \ldots, M_{R_1}\}$. The cumulative distribution of the largest value M^1 in this set is then equal to

$$\Pr\{M^1 < m\} = \langle H(m-M^1)\rangle = \left\langle \prod_{i=1}^{r} H(m-M_i)\right\rangle. \tag{10}$$

The interest in $\Theta_1(z,m|\rho)$ in (8),(9) lies in particular in the fact that, for $z=1$, it reduces to the probability $P_1(m|\rho)$ that the largest magnitude among all aftershocks of the first generation is smaller than m:

$$P_1(m|\rho) = \Theta_1(z=1,m|\rho) = \Pr\{M_\rho^1 < m\} = e^{-\kappa\mu(\rho)\mathcal{Q}(m)}. \tag{11}$$

B. GPF of aftershocks of the first generation triggered by a spontaneous source

Consider now some spontaneous source [contributing to the term $s(t,\mathbf{r})$ in (5)], with magnitude M_0. According to the ETAS model, it triggers its own aftershocks sequence independently of all other sequences. Let

$$M_0,\ M_1,\ M_2,\ \ldots, M_{R^1} \tag{12}$$

be the random sequence of magnitudes (including M_0) of the first-generation aftershocks triggered by the spontaneous source. The GPF

$$\Theta_1(z,m) = \langle z^{R^1(m)} H(m - M^1)\rangle, \tag{13}$$

describing the statistics of the number $R^1(m)$ of aftershocks of first generation (triggered by the spontaneous source of arbitrary magnitude M_0) and their largest magnitude M^1 among the list (12), can then be calculated following a formula analogous to (8). Specifically, we multiply both sides of (8) by $H(m-\rho)$ and apply the statistical averaging, assuming that ρ (which in this context is the random magnitude M_0 of the spontaneous source) is a random variable distributed according to the Gutenberg-Richter law (3). In other words, we calculate the GPF $\Theta_1(z,m)$ using the relation

$$\Theta_1(z,m) = \langle H(m-\rho)\Theta_1(z,m|\rho)\rangle. \tag{14}$$

Here, the angular brackets indicates the statistical average with respect to the random magnitude ρ of the spontaneous source. Using (14) together with the right-hand side (r.h.s.) of Eq. (8), we obtain

$$\Theta_1(z,m) = \int_{m_0}^{m} e^{\kappa\mu(\rho)[z\mathcal{P}(m)-1]} p(\rho) d\rho. \tag{15}$$

In order to calculate the integral, it is convenient to introduce the auxiliary function

$$\mathcal{F}(y,x) = \int_x^\infty e^{-\kappa\mu(\rho)y} p(\rho) d\rho = -\int_x^\infty e^{-\kappa\mu(\rho)y} d\mathcal{Q}(\rho), \tag{16}$$

such that

$$\Theta_1(z,m) = \mathcal{F}(1 - z\mathcal{P}(m), m_0) - \mathcal{F}(1 - z\mathcal{P}(m), m). \tag{17}$$

It follows from (4) and (16) that

$$\mathcal{F}(y,x) = \gamma \int_{\mu(x)}^\infty e^{-\kappa\mu y} \mu^{-\gamma-1} d\mu = \mu^{-\gamma}(x) F(\mu(x) y), \tag{18}$$

where

$$F(y) = \gamma \int_1^\infty e^{-\kappa\mu y} \mu^{-\gamma-1} d\mu = \gamma\kappa^\gamma y^\gamma \Gamma(-\gamma, \kappa y). \tag{19}$$

Substituting (18) into (17) and taking into account that $\mu(m_0)=1$ yields the following expression:

$$\Theta_1(z,m) = F(1 - z\mathcal{P}(m)) - [\mu(m)]^{-\gamma} F(\mu(m)[1 - z\mathcal{P}(m)]). \tag{20}$$

For $m \to \infty$, the GPF $\Theta_1(z,m)$ given by (20) reduces to the standard GPF of the random number R_1 of aftershocks of the first generation triggered by some spontaneous source, which reads

$$\Theta_1(z) = \Theta_1(z, m=\infty) = F(1-z). \tag{21}$$

In the following analysis, the function $F(y)$ in (19) plays a crucial role. It is thus useful to state some of its analytical properties. Recall that $\Gamma(-\gamma,x)$ in (19) is the incomplete Gamma function, defined by $\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} dt$. Using the Taylor series expansion [5]

$$\Gamma(a,x) = \Gamma(\alpha) - \sum_{i=0}^\infty \frac{(-1)^i x^{a+i}}{i!(a+i)} \tag{22}$$

yields

$$F(y) = \gamma\Gamma(-\gamma)(\kappa y)^\gamma + 1 - \frac{\gamma}{\gamma-1}\kappa y - \frac{\gamma}{2(2-\gamma)}(\kappa y)^2 + \cdots. \tag{23}$$

In the theory of aftershocks branching processes, the branching ratio n, equal to the average number of first-generation aftershocks triggered by some mother earthquake, plays a fundamental role since it controls the subcriticality vs supercriticality of the process. It is defined as

$$n = \left.\frac{d\Theta_1(z)}{dz}\right|_{z=1}.$$

Together with (21), it follows that

$$n = -\left.\frac{dF(y)}{dy}\right|_{y=0} = \frac{\kappa\gamma}{\gamma-1}, \tag{24}$$

so that, for given b, α, and n, the constant κ in relation (23) can be replaced by

$$\kappa = \kappa(\gamma, n) = n\left(1 - \frac{1}{\gamma}\right). \tag{25}$$

We thus obtain the following first terms in the power expansion of $F(y)$:

$$F(y) \simeq 1 - ny + \beta y^\gamma - dy^2, \tag{26}$$

where

$$\beta = \gamma\Gamma(-\gamma)\left(n\frac{\gamma-1}{\gamma}\right)^\gamma, \quad d = n^2 \frac{(\gamma-1)^2}{2\gamma(2-\gamma)}. \tag{27}$$

In our paper, we use the truncated relation

$$F(y) \simeq 1 - ny + \beta y^\gamma, \tag{28}$$

which provides an accurate description of $F(y)$ for $1 < \gamma \lesssim 1.5$ and $y \lesssim 0.2$.

Taking $z=1$ in (20) yields the cumulative distribution function (CDF) $P_1(m)$ of the largest magnitude M^1 among the sequence (12) of aftershocks, including their spontaneous source

$$P_1 = F(\mathcal{Q}) - \mu^{-\gamma} F(\mu \mathcal{Q}). \quad (29)$$

The GPF $\Theta_1(z,m)$ in (20) describes the statistics of the spontaneous source and its first-generation aftershocks, such that all magnitudes, including the magnitude of the spontaneous source, are smaller than m. In the following, we will consider the possibility that the largest aftershock may be larger than the source, a situation known in the seismological literature as the occurrence of foreshocks (see [3,17,18] and references therein). The corresponding GPF, averaged over all possible foreshock magnitudes, is denoted as $\overline{\theta_1}(z,m)$ and is obtained formally from (20) by taking the limit $\mu \to \infty$:

$$\overline{\theta_1}(z,m) = F(1 - z\mathcal{P}(m)). \quad (30)$$

C. Magnitude of the largest aftershock of the first-generation aftershocks

Before analyzing the conditions under which the empirical Båth's law can be obtained from the ETAS model, it is useful to ask what is its analog when restricting the set of aftershocks to the first generation of events triggered by the source. This will provide a reference point against which to gauge the impact of the multiple generations of aftershocks on Båth's law.

We start from expression (11) giving the CDF $P_1(m|\rho)$ of the largest magnitude M_ρ^1 among all aftershocks of the first generation. Substituting (2),(4) in (11), we obtain

$$P_1(m|\rho) = G[w_0(m - \rho + v_0)], \quad (31)$$

where

$$v_0(\rho) = \left(1 - \frac{\alpha}{b}\right)(\rho - m_0) + \frac{1}{b}\log_{10}\left(\frac{b}{n(b-\alpha)}\right),$$

$$w_0 = b \ln 10, \quad (32)$$

and

$$G(x) = \exp(-e^{-x}) \quad (33)$$

is the well-known limiting extremal Gumbel CDF.

It follows from (31) in particular that the probability density function (PDF) of the difference

$$\Delta_\rho^1 = \rho - M_\rho^1 \quad (34)$$

between the source (mainshock) magnitude and the magnitude of its largest aftershock of the first generation is equal to

$$f_1(\delta|\rho) = w_0 g(w_0(v_0 - \delta)), \quad (35)$$

where

$$g(x) = \exp(-x - e^{-x}) \quad (36)$$

is the PDF associated with the CDF (33). Note that the shape and variance of the PDF (35) does not depend on the mainshock magnitude ρ. Only its mode $v_0(\rho)$ [most probable value of the difference (34)] depends on ρ and increases linearly with it according to the first equation of (32).

These results treat all aftershock sequences on the same footing and in particular include sequences that have zero aftershocks. In a real data analysis, the statistical properties of the difference (34) are obtained conditional on the observation of at least one aftershock, which requires a modification of the expressions above. We are interested in modifying the CDF (11) to eliminate the cases where $R_\rho^1 = 0$. This corresponds to obtaining the CDF of the largest magnitude of first-generation aftershocks under the condition that the mainshock triggers at least one aftershock:

$$P_1(m|\rho;1) = 1 - Q_1(m|\rho;1), \quad Q_1(m|\rho;1) = \frac{1 - e^{-\kappa\mu(\rho)\mathcal{Q}(m)}}{1 - e^{-\kappa\mu(\rho)}}. \quad (37)$$

The corresponding PDF of the difference (34) reads

$$f_1(\delta|\rho) = w_0 \frac{g(w_0(v_0 - \delta))}{1 - e^{-\kappa\mu(\rho)}}, \quad -\infty < \delta < \rho - m_0. \quad (38)$$

The conditional CDF (37) differs significantly from the unconditional one (11) only if the probability $e^{-\kappa\mu(\rho)}$ that there are no aftershocks is close to 1. This occurs for small mainshock magnitudes. How small should the mainshocks be for this difference to be important? Let us define a magnitude threshold ρ_0 by

$$\kappa\mu(\rho) \simeq 2 \Rightarrow \rho_0 = m_0 + \frac{1}{\alpha}\log_{10}\left(\frac{2b}{n(b-\alpha)}\right). \quad (39)$$

For mainshock magnitudes $\rho > \rho_0$, the conditional CDF (37) does not differ significantly from the unconditional one (11). In this case, relations (35) and (38) are approximately equal, and the looked for distribution of the difference (34) can be taken to be the PDF (35) where $\delta \in (-\infty, \infty)$. Thus, for $\rho > \rho_0$, the average of the magnitude difference (34) can be approximated by

$$\Delta_\rho^1 m \equiv \rho - \langle M_\rho^1 \rangle \simeq w_0 \int_{-\infty}^{\infty} \delta g(w_0(v_0 - \delta)) d\delta = v_0(\rho) - \nu/w_0, \quad (40)$$

where $\nu \simeq 0.5772$ is the Euler constant. Figure 1 shows the exact average difference $\Delta_\rho^1 m$ calculated with (38), its approximation (40) valid for sufficiently large mainshocks $\rho > \rho_0$ and the most probable value $\Delta_\rho^1 m_* = v_0(\rho)$ of the difference in magnitude between the mainshock magnitude ρ and its largest aftershock. This figure is typical of the strong dependence found for all reasonable values of the parameters and distinguishes this result from the empirical Båth's law (which gives a constant value independent of ρ).

IV. STATISTICS OF THE LARGEST AFTERSHOCK AMONG AFTERSHOCKS OF ALL GENERATIONS

A. GPF of the aftershocks over all generations triggered by a spontaneous source

Due to the mutual statistical independence of different branches of triggered earthquakes in the ETAS model, one can easily generalize the results for the largest aftershock of the first generation to derive the statistical properties of the largest aftershock over all generations.

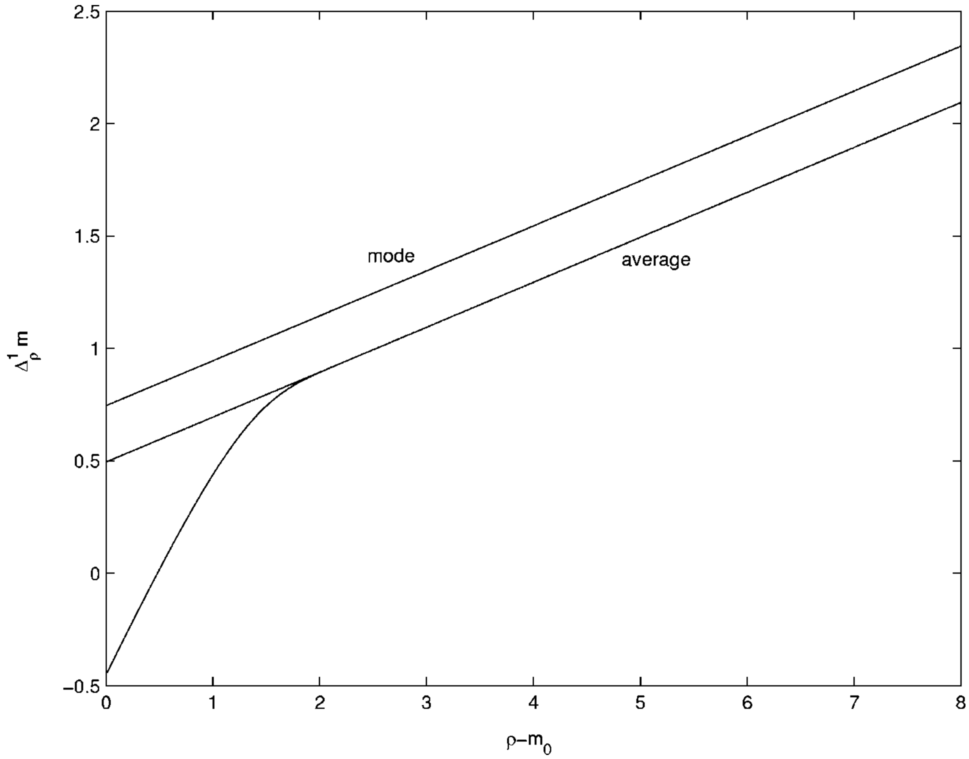

FIG. 1. Exact average of the difference $\Delta_1 m = \rho - \langle M_\rho^1 \rangle$ obtained using (38) (bottom curve bending down for small $\rho - m_0$), its large magnitude approximation (40) (bottom straight line) and the difference $\Delta_\rho^1 m_* = v_0(\rho)$ between the mainshock magnitude ρ and the mode of the magnitude of the largest aftershock among all aftershocks of the first generation (upper straight line), for $n=0.9$, $\alpha=0.8$, and $b=1$.

Within the ETAS branching model, taking into account all aftershocks of all generations triggered by the mainshock, amounts to replacing in the r.h.s. of Eq. (8) the PDF $\mathcal{P}(m)$ of the magnitudes of single aftershocks by the GPF $\Theta(z,m)$ for all aftershocks triggered by some spontaneous source that have (together with the source) magnitudes smaller than m. As a result, we obtain the sought GPF of the number of all aftershocks triggered by the mainshock conditional on all magnitudes being less than m as

$$\Theta(z,m|\rho) = \langle z^{R_\rho(m)} H(m - M_\rho) \rangle = e^{\kappa\mu(\rho)[z\Theta(z,m)-1]}. \quad (41)$$

Here, M_ρ is the magnitude of the largest aftershock. In particular, the complementary CDF of the magnitude of the largest aftershock reads

$$Q(m|\rho) = \langle H(M_\rho - m) \rangle = 1 - e^{-\kappa\mu(\rho)Q(m)}, \quad (42)$$

where Q is defined in (45).

Similarly, the functional equation for the GPF $\Theta(z,m)$ is obtained by replacing in (20) both Θ_1 and $\mathcal{P}(m)$ by $\Theta(z,m)$:

$$\Theta(z,m) = F(1 - z\Theta(z,m)) - \mu^{-\gamma}(m)F(\mu(m)[1 - z\Theta(z,m)]). \quad (43)$$

For $z=1$, Eq. (43) reduces to an equation for the CDF $P(m)$ of the magnitude M of the largest event (including all aftershocks and the source) defined as

$$\Theta(z=1,m) = P(m) = \Pr\{M < m\}. \quad (44)$$

This equation reads

$$P = F(Q) - \mu^{-\gamma}F(\mu Q), \quad Q(m) = 1 - P(m). \quad (45)$$

The difference between (45) and (29) is that (45) is an implicit equation for P while, in (29), P_1 is an explicit function of the Q defined in (4). Notice that in the limit $m \to \infty$, Eq. (43) reduces to the well-known functional equation

$$\Theta = \Theta_1(z\Theta) \quad (46)$$

for the standard GPF

$$\Theta(z,m=\infty) = \Theta(z) = \langle z^R \rangle \quad (47)$$

of the random number R of all aftershocks triggered by some ancestor, which has been studied in [6,8].

Similar to the reasoning leading to (37), the conditional probability that the magnitude of the largest aftershock exceeds m, under the condition that the mainshock triggers at least one aftershock, is

$$Q(m|\rho;1) = \frac{1 - e^{-\kappa\mu(\rho)Q(m)}}{1 - e^{-\kappa\mu(\rho)}}, \quad Q(m) = 1 - P(m). \quad (48)$$

It is also of interest to obtain the GPF $\overline{\Theta}(z,m)$ of the number of aftershocks of all generations with magnitudes smaller than m that are triggered by some spontaneous source of arbitrary magnitude. It is given by replacing in the r.h.s. of Eq. (30) $\mathcal{P}(m)$ by the GPF $\Theta(z,m)$ given by (43), which yields

$$\overline{\Theta}(z,m) = F(1 - z\Theta(z,m)). \quad (49)$$

For $z=1$, this gives the probability $\overline{P}(m)$ that the magnitude of the largest aftershock triggered by an arbitrary spontaneous source is smaller than m:

$$\overline{P}(m) = F(Q(m)). \quad (50)$$

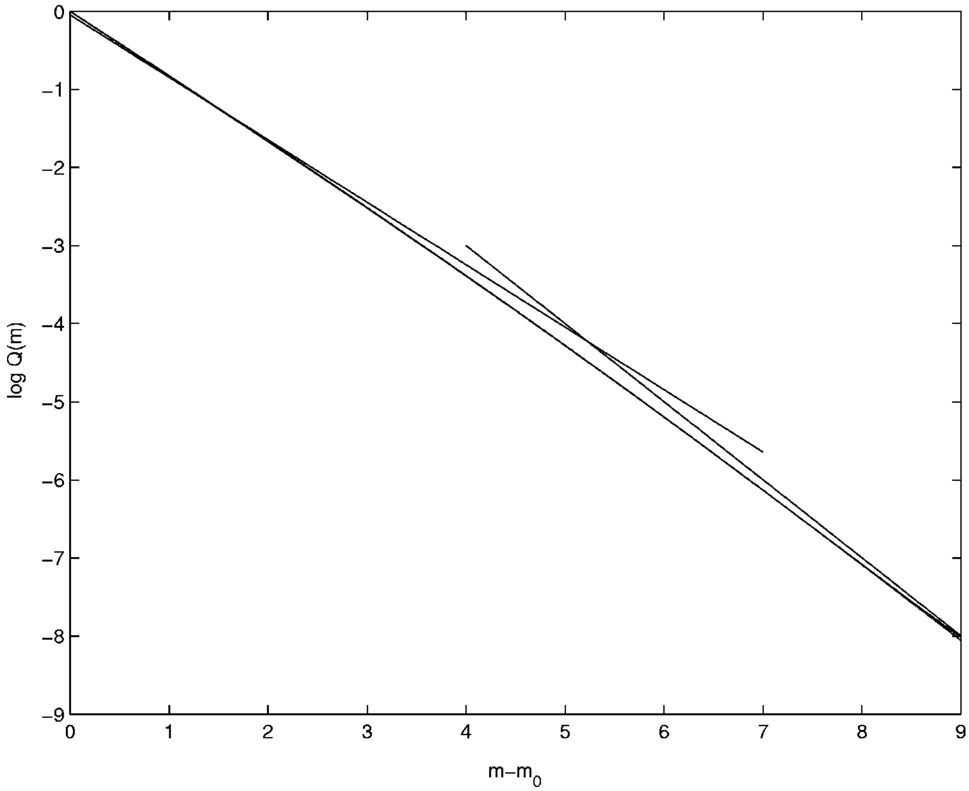

FIG. 2. Plot of the decimal logarithm of the exact CDF $Q(m)$ and its approximation (51) (which actually coincide) for $n=0.9$, $\alpha=0.8$, and $b=1$. Straight lines correspond to the asymptotics (53) and (54).

B. Distribution of the magnitude of the largest aftershock of a spontaneous earthquake source of arbitrary magnitude

All quantities defined above require the knowledge of $Q(m)$, which has a straightforward statistical meaning: it is the complementary CDF of the magnitude of the largest aftershock of a spontaneous source of arbitrary magnitude (in other words, over all possible source magnitudes). It is easy to calculate $Q(m)$ by solving Eq. (45) numerically. We can also use the algebraic approximation (28) of the function $F(y)$ to obtain an explicit and rather precise analytic expression of $Q(m)$. Indeed, substituting (28) into (45) yields

$$Q(m) \simeq \frac{1}{(1-n)[\mu(m)]^{\gamma} + n\mu(m)}. \tag{51}$$

This expression shows that there is a crossover magnitude m_c, given by

$$\mu(m_c) \simeq \left(\frac{n}{1-n}\right)^{1/(\gamma-1)} \Rightarrow m_c \simeq m_0 + \frac{1}{b-\alpha}\log_{10}\left(\frac{n}{1-n}\right), \tag{52}$$

separating two regimes with different power laws for $Q(m)$. The first regime

$$Q(m) \simeq \frac{1}{n}10^{-\alpha(m-m_0)}, \quad m \lesssim m_c, \tag{53}$$

corresponds to a complementary CDF decaying more slowly than the Gutenberg-Richter law (3),(4), for $\alpha < b$. In the critical case $n=1$, $m_c=\infty$ and this regime (53) holds for any $m > m_0$. The second regime recovers the Gutenberg-Richter law

$$Q(m) \simeq \frac{1}{1-n}10^{-b(m-m_0)}, \quad m \gtrsim m_c. \tag{54}$$

Figure 2 shows the logarithm (in base 10) of the complementary CDF $Q(m)$ as a function of $m-m_0$ and the two power law asymptotics (53) and (54). We have thus shown that the Gutenberg-Richter law can be renormalized from a bare exponent b to a smaller exponent α when the distribution is restricted to the set of largest aftershocks of spontaneous earthquakes of arbitrary magnitudes. This renormalization of the b value from b to α is intrinsically a cascade phenomenon. In other words, it results from the existence of a cascade of triggered earthquakes over many generations, as shown in detail in [6]. This renormalization proceeds by a mechanism similar to that of the p value of the Omori law from a value $1+\theta$ to $1-\theta$ [1,2]. It is different from the mechanism leading to an exponent $b-\alpha$ for the asymptotic branch of the Gutenberg-Richter distribution of all foreshocks [18].

This prediction (53) offers a method for measuring the key exponent α controlling the productivity or triggering efficiency of earthquakes as a function of their magnitude, according to (2). What is needed to implement this method is a declustering technique to identify the spontaneous sources and their largest aftershocks. The statistical declustering technique of Zhuang *et al.* [19,20] seems to be particularly suitable for this purpose.

Due to the independence between the aftershock sequences of different sources in the ETAS model, it is straightforward to obtain the probability distribution of the largest triggered events among a set of r spontaneous sources. The corresponding complementary CDF, giving the

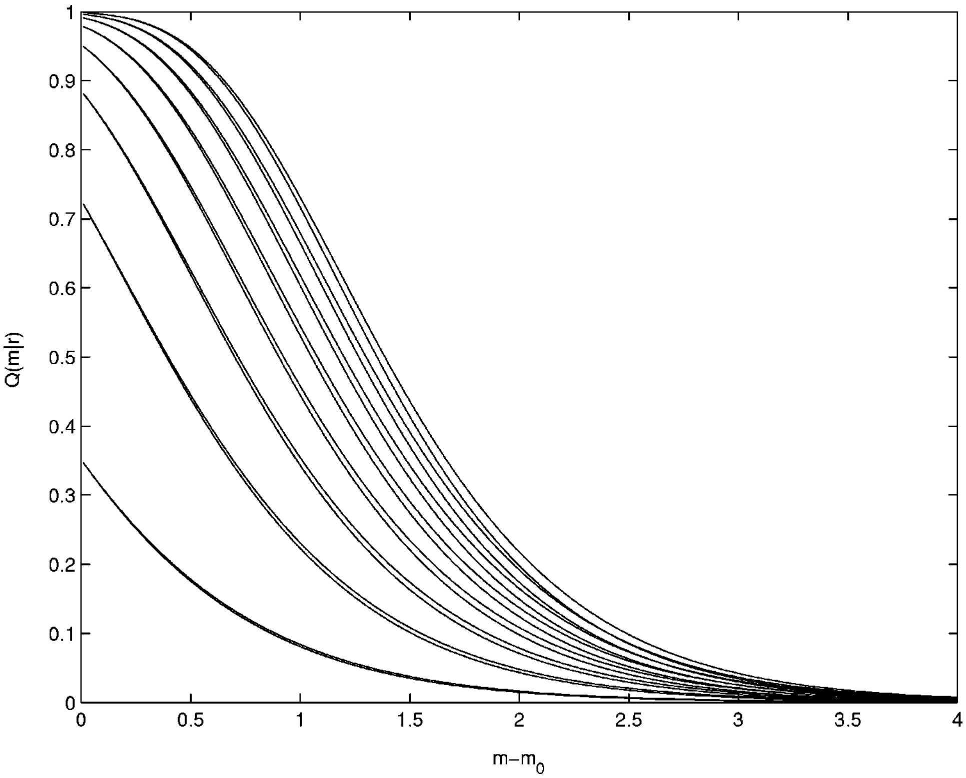

FIG. 3. Complementary CDF (55) of the magnitude of the largest event triggered by r spontaneous sources with random magnitudes chosen according to the Gutenberg-Richter distribution. The different curves correspond to $r=1,3,5,7,9,11,13,15$ from bottom to top. Each pair of curves corresponds to $n=0.9$ and $n=1$, respectively.

probability that the largest event triggered over all generations by r sources is larger than m, is equal to

$$\overline{Q}(m|r) = 1 - F^r(Q(m)). \qquad (55)$$

Figure 3 plots $\overline{Q}(m|r)$ as a function of $m-m_0$, for r running from 1 to 15 for $\alpha=0.8$, $b=1$ and for two values of n: $n=0.9$ and $n=1$. It is clear that most of the largest triggered events in aftershock sequences are very small, simply due to the interplay of two factors: most random sources are themselves small and have a small productivity, and the Gutenberg-Richter distribution makes it much more probable that all triggered events have small magnitudes.

C. Distribution of the magnitude of the largest aftershock of a spontaneous earthquake source of fixed magnitude ρ

Rather than considering arbitrary source magnitudes, it is interesting to determine the complementary CDF $Q(m|\rho,r)$ of the magnitude of the largest event triggered by r spontaneous sources with fixed magnitudes $\rho_1,\rho_2,\ldots,\rho_r$. It is closely approximated by

$$Q(m|\rho,r) \simeq 1 - \exp\left[-\kappa Q(m)\sum_{i=1}^{r}\mu(\rho_i)\right]. \qquad (56)$$

In the following, we restrict our analysis to the case of a single $r=1$ spontaneous source which fixed magnitude ρ. In this case, expression (56) transforms into (42).

For $m<m_c$, where the crossover magnitude m_c is defined by (52), expression (42) can be simplified by replacing the exact $Q(m)$ by the approximation (53), which gives

$$Q(m|\rho) \simeq 1 - G(w_1(m-\rho+v_1)), \quad m_0 < m \lesssim m_c. \qquad (57)$$

where

$$v_1 = \frac{1}{\alpha}\log_{10}\left(\frac{b}{b-\alpha}\right), \quad w_1 = \alpha \ln 10. \qquad (58)$$

Expression (57) can be further simplified into

$$Q(m|\rho) \simeq \frac{\kappa}{n}10^{-\alpha(m-\rho)} = (\kappa 10^{\alpha(\rho-m_0)}) \times \left(\frac{1}{n}10^{-\alpha(m-m_0)}\right), \qquad (59)$$

in the tail of $Q(m|\rho)$, i.e., for $\rho-(1/\alpha)\log_{10}(n/\kappa)<m$. The rewriting of $Q(m|\rho)$ under the form shown by the last equality in (59) clarifies its origin: the first factor $\kappa\, 10^{\alpha(\rho-m_0)}$ is nothing but the productivity law (2); the second factor $(1/n)10^{-\alpha(m-m_0)}$ is the renormalized Gutenberg-Richter law (53).

For $m>m_c$, we obtain another approximation for $Q(m|\rho)$ by replacing in the r.h.s. of Eq. (42) the complementary CDF $Q(m)$ by the approximation (54), which yields

$$Q(m|\rho) \simeq 1 - G(w_2[m-\rho+v_2(\rho)]) \quad m \gtrsim m_c, \qquad (60)$$

where

$$v_2 = \left(1-\frac{\alpha}{b}\right)(\rho-m_0) + \frac{1}{b}\log_{10}\left(\frac{b(1-n)}{n(b-\alpha)}\right), \quad w_2 = b\ln 10. \qquad (61)$$

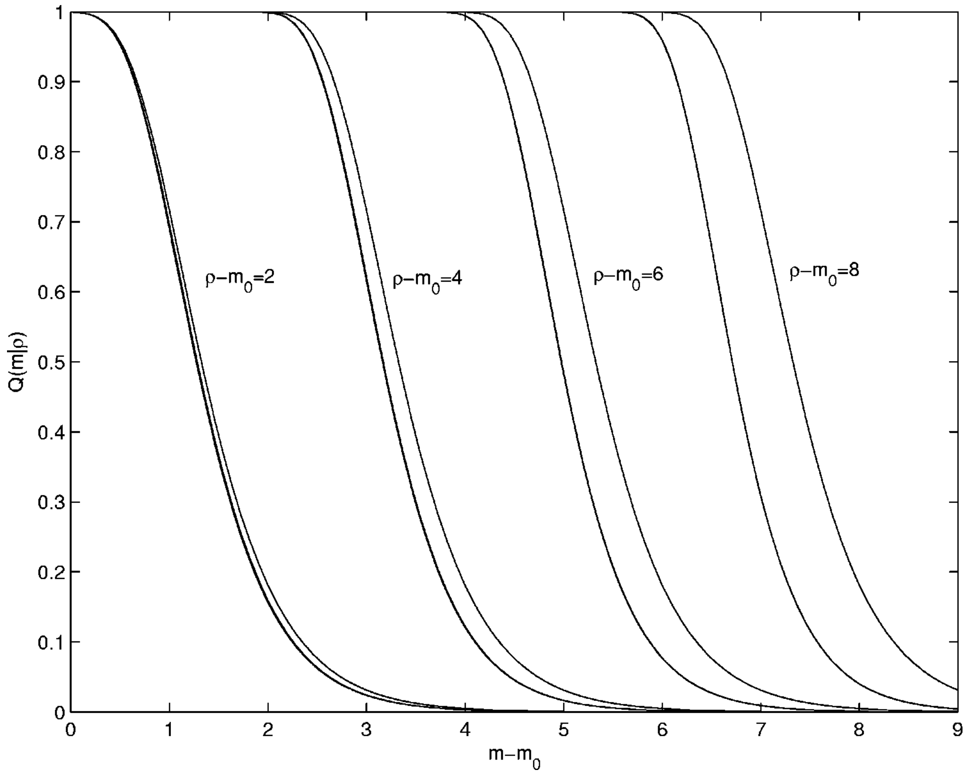

FIG. 4. Exact complementary CDF's $Q(m|\rho)$ given by (42) (lower curves), their approximations (62) (which are actually undistinguishable from the exact functions) and the universal approximations (57) (upper curves), for $n=0.9$, $b=1$, and $\alpha=0.8$ ($\gamma \equiv b/\alpha=1.25$) for four different values of the spontaneous source magnitude ρ.

The following approximation including both regimes $m < m_c$, $m > m_c$ and laws (57),(60) is obtained from (42) by replacing $Q(m)$ by the approximation (51)

$$Q(m|\rho) \simeq 1 - \exp\left(-\frac{\kappa\mu(\rho)}{(1-n)\mu^{\gamma}(m) + n\mu(m)}\right). \quad (62)$$

Figures 4 and 5 present the dependence of the complementary CDF $Q(m|\rho)$ as a function of the magnitude of the largest event triggered by a spontaneous source of fixed magnitude ρ, for four different values of ρ. The figures show the exact $Q(m|\rho)$ obtained numerically, its approximation (62) (which is actually undistinguishable from the exact one), and

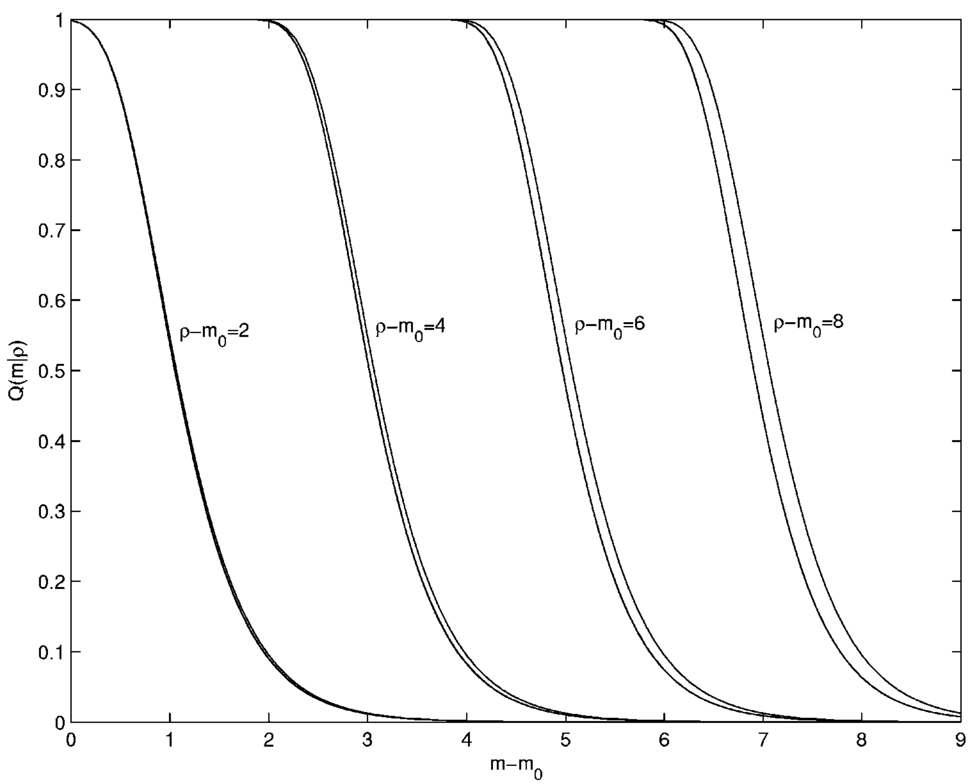

FIG. 5. Same as Fig. 4 for $\alpha=0.9$ ($\gamma=1.11$).

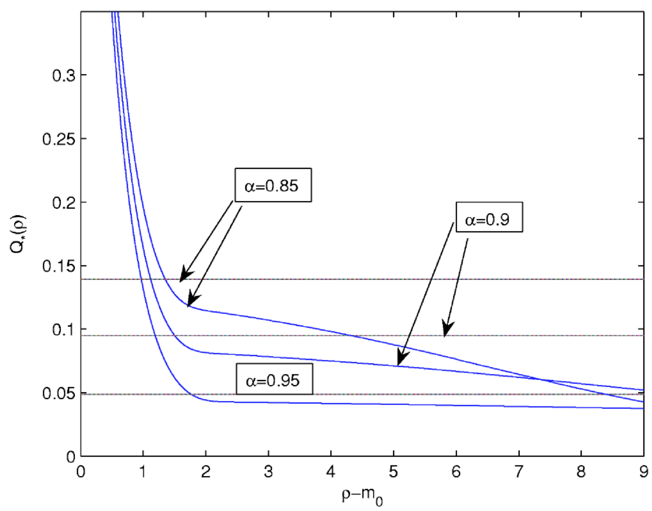

FIG. 6. Plot of $Q_*(\rho)$ as a function of ρ, for $n=0.9$, $b=1$, and different values of α. The horizontal lines correspond to the constants predicted by (64).

the universal approximation (57). The comparison between Fig. 4 (for $\alpha=0.8$) and Fig. 5 (for $\alpha=0.9$) shows that the approximation (57) becomes more and more precise as α approaches b.

As a bonus, we obtain the probability $Q_*(\rho)$ that the magnitude m of the largest aftershock exceeds the source magnitude ρ, a situation that is usually classified in seismic catalogs by saying that the largest triggered event is the mainshock and the spontaneous source that initiated the sequence and all triggered events before the largest aftershock are foreshocks. Indeed, $Q_*(\rho)$ is nothing but

$$Q_*(\rho) \equiv Q(\rho|\rho;1). \tag{63}$$

Interestingly, in the regime $m_0 < \rho < m_c$ (i.e., for n sufficiently close to 1 and/or α close to b) for which (57) holds, we obtain

$$Q_*(\rho) \simeq \text{const} = Q_* = 1 - \exp\left(\frac{\alpha - b}{b}\right), \tag{64}$$

which is independent of ρ and of the branching ratio n. This approximation is all the better, the closer α is to b. Figure 6 shows the exact $Q_*(\rho)$ as a function of ρ, which can be compared with the constant (64). One can observe that, at least for $\alpha=0.95$, $Q_*(\rho)$ is actually quite close to the constant (64) over all possible magnitudes $\rho > m_0$. We propose to call the prediction (64) the second Båth law for foreshocks: the probability that a main earthquake turns out to be the foreshock does not depend on its magnitude ρ (more generally, the distribution of the difference $\rho - m$ does not depend on ρ).

V. DERIVATION OF BÅTH'S LAW

The derivation of the distribution (48) and the approximations (57) and (60) allow us to derive Båth's law by calculating the statistical average

$$\Delta_\rho m = \rho - \langle M_\rho \rangle, \tag{65}$$

where M_ρ denotes the magnitude of the largest aftershock among all events of all generations triggered by the source of fixed magnitude ρ. Recall that Båth's law states that $\Delta_\rho m$ is independent of ρ and equal to 1.2.

Within the ETAS model, the exact value of $\Delta_\rho m$ is obtained as

$$\Delta_\rho m = \rho - \int_{m_0}^{\infty} Q(m|\rho;1)\, dm, \tag{66}$$

where $Q(m|\rho;1)$ is given by (48) and the r.h.s. of (66) expresses the fact that the average is performed over sequences with at least one aftershock.

The two regimes—$m < m_c$ giving the asymptotic (57) and $m > m_c$ giving the asymptotic (60)—provide two asymptotic expressions for $\Delta_\rho m$. Indeed, calculating $\langle M_\rho \rangle$ using the approximation (57) and neglecting the boundary effects [i.e., supposing that $m \in (-\infty, \infty)$] yields

$$\Delta_\rho m_1 \simeq v_1 - \frac{\nu}{w_1} = B = \frac{1}{\alpha}\left[\log_{10}\left(\frac{b}{b-\alpha}\right) - \frac{\nu}{\ln 10}\right], \tag{67}$$

which is independent of ρ. Recall that $\nu \simeq 0.5772$ is Euler's constant. In the following, we designate B defined in (67) as Båth's constant. Note that this regime $m < m_c$ corresponds to the critical branching regime of n close to 1 (for a fixed magnitude ρ) and expresses the full effect of the cascade of triggered events over all possible generations. It is remarkable that the theory of the ETAS branching model predicts the first part of Båth's law that the average of the difference between the magnitude of a mainshock and its largest aftershock is independent of the mainshock magnitude. The specific value of Båth's constant B depends on only two parameters: the b value of the Gutenberg-Richter distribution and the productivity exponent α.

The second asymptotic for $m > m_c$ corresponds to using (60) to estimate $\langle M_\rho \rangle$, which yields

$$\Delta_\rho m_2 \simeq v_2 - \frac{\nu}{w_2} = \left(1 - \frac{\alpha}{b}\right)(\rho - m_0) + \frac{1}{b}\left[\log_{10}\left(\frac{(1-n)b}{n(b-\alpha)}\right) - \frac{\nu}{\ln 10}\right]. \tag{68}$$

Note that $\Delta_\rho m_2$ is increasing with ρ as in expression (40), corresponding to taking into account aftershocks of only the first generation. This is natural since the asymptotic for $m > m_c$ corresponds to n relatively far from 1 (for a fixed ρ), i.e., far from the critical branching regime, such that only a few generations play a significant role in the population of aftershocks. This asymptotic (68) is also identical to the expression (5) of [14] derived by using the statistical average of the total number N_{aft} of aftershocks of all generations triggered by a source of fixed given magnitude. Thus, the difference between this approximation (68) [and expression (5) of [14]] and the exact expression (66) and its critical universal asymptotic (67) can be traced back to the difference between the following two kinds of averages: $\langle \ln[N_{\text{aft}}] \rangle$ and $\ln\langle N_{\text{aft}} \rangle$.

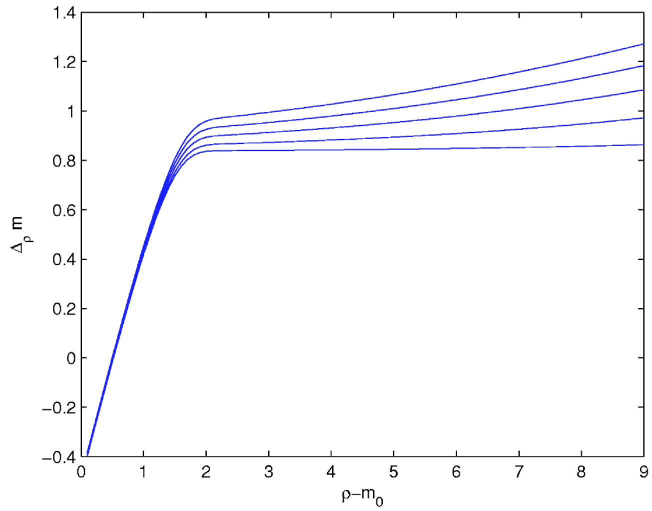

FIG. 7. Plot of $\Delta_\rho m$ given by the exact expression (66) for $\alpha=0.9$, $b=1$, and for different branching ratio. Top to bottom $n=0.8, 0.85, 0.9, 0.95, 0.99$. With n tending to 1, the average magnitude difference become closer to the theoretical Båth constant $B=0.83$.

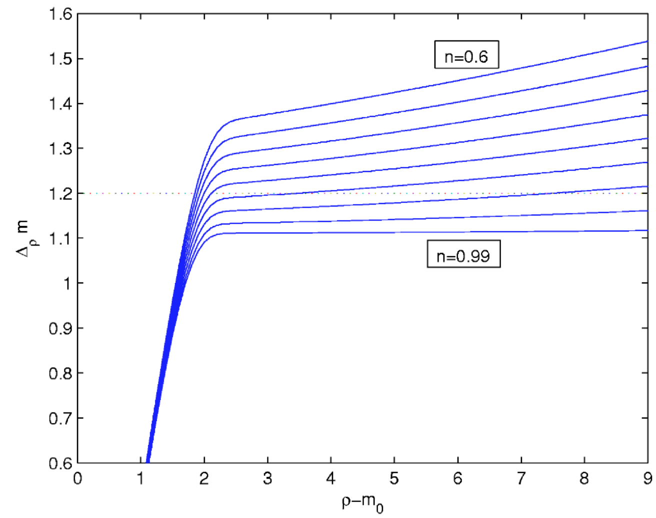

FIG. 8. Same as Fig. 7 (except for the magnification) for $\alpha=0.95$ ($\gamma\simeq 1.05$) giving $B=1.11$.

Figures 7 and 8 show the exact average magnitude difference (66) as a function of the mainshock magnitude ρ for $b=1$ and different values of the branching ratio n, for $\alpha=0.9$ (Fig. 7) and $\alpha=0.95$ (Fig. 8). As expected from the condition $m<m_c$ or n closer to 1 at fixed ρ so that m_c is all the larger according to (52), the largest values of n give almost constant values of $\Delta_\rho m$, in agreement with the prediction (67). For smaller n's, we can observe a slow crossover to the second asymptotic (68). By comparison of Figs. 7 and 8, it is clear that low values of α are not compatible with Båth's law. As confirmed with similar figures obtained for smaller α's, a value of α at least equal to 0.9 seems necessary to obtain a dependence of $\Delta_\rho m$ roughly independent of ρ over a large magnitude range. This bound is compatible with some previous studies [15,16,21], but is in disagreement with others [20,22]. However, we should remark that such heterogeneity in reported values of key parameters of the ETAS model such as the productivity exponent α could be due to the bias resulting from imperfect account of unobserved seismicity below the completeness threshold, which may play a dominant role as explained in [23,24]. Figure 8 also shows that, the closer n is to 1, the more independent is $\Delta_\rho m$ with respect to the mainshock magnitude ρ. However, due to inherent fluctuations in empirical data, n can be as low as $n=0.8$ for $\alpha\simeq 0.95$ and $\Delta_\rho m$ would still be slowly growing between 1.1 and 1.3 over a large magnitude range of ρ, so that Båth's law would be approximately verified.

VI. CONCLUDING REMARKS

Using the ETAS branching model of triggered seismicity, we have shown how to calculate exactly the average difference between the magnitude of a mainshock and the magnitude of its largest aftershock over all generations. This average magnitude difference is found empirically to be independent of the mainshock magnitude and equal to 1.2, a universal behavior known as Båth's law. We have developed the mathematical formulation in terms of generating probability functions that allow us to obtain exact equations and useful approximations to understand the physical basis for Båth's law. In particular, we find that the constancy of the average magnitude difference (to a value that we term Båth's constant) is associated with the critical regime of the ETAS branching process. Allowing for error bars ±0.1 for Båth's constant value around 1.2, our exact analytical treatment of Båth's law provides a new constraint on two key parameters of the ETAS model; namely, the productivity exponent α and the branching ratio n: $\alpha\gtrsim 0.9$ and $n\gtrsim 0.8$. We have suggested a method for measuring α based on the predicted renormalization of the Gutenberg-Richter distribution of the magnitudes of the largest aftershock. To implement this method, statistical declustering techniques can be used to identify the spontaneous sources and their largest aftershocks. We have also proposed the "second Båth law for foreshocks" that the probability that a main earthquake turns out to be the foreshock does not depend on its magnitude ρ.

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